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ABSOLUTELY COUNTALY COMPACT SPACES AND RELATED SPACES

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§1. INTRODUCTION

By a space, we mean a topological space. Matveev [7] defined a space X to be *absolutely countably compact* (= *acc*) if for every open cover $\mathcal{U}$ of X and every dense subspace $D \subset X$, there exists a finite subset $F \subset D$ such that $\mathrm{St}(F, \mathcal{U}) = X$ and defined a space X to be *hereditarily absolutely countably compact* (= *hacc*) if all closed subspaces of X are acc. In [8], he also defined a space X to have the *property* (a) (resp. *property* (wa)) if for every open cover $\mathcal{U}$ of X and every dense subspace D of X, there exists a discrete closed subspace (resp. discrete subspace) $F \subset D$ such that $\mathrm{St}(F, \mathcal{U}) = X$. By the definitions, all compact spaces are hacc, all hacc spaces are acc, all acc spaces have the property (a) and all spaces having the property (a) have the property (wa). Moreover, it is known [7] that all acc spaces are countably compact (cf. also [4]). Thus, we have the following diagram:

$$\begin{tikzcd}
\text{compact} \arrow[r] & \text{hacc} \arrow[r] & \text{acc} \arrow[r, "T_2"] \arrow[d] & \text{countably compact} \\
 & & \text{property } (a) \arrow[r] & \text{property } (wa)
\end{tikzcd}$$

In the above diagram, the converse of each arrow does not hold, in general (cf. [7], [8]). For an infinite cardinality κ, a space X is called *initially* κ*-compact* if every open cover of X with cardinality $\leq \kappa$ has a finite subcover. The main theorems of this paper are Theorems 1, 2 and 3 below. We prove only Theorem 2 here and leave the details of the proofs of Theorems 1 and 3 to elsewhere.

Theorem 1. *Let κ be an infinite cardinal. Let X be an initially κ-compact T_3-space, Y a compact T_2-space with $t(Y) \leq \kappa$ and A a closed subspace of $X \times Y$. Assume that $A \cap (X \times \{y\})$ is acc for each $y \in Y$ and the projection $\pi_Y : X \times Y \to Y$ is a closed map. Then, the subspace A is acc.*

Vaughan [12] proved that

(i) if X is an acc T_3-space and Y is a sequential, compact T_2-space, then $X \times Y$ is acc, and

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(ii) if X is an ω-bounded, acc T_3-space and Y is a compact T_2-space with $t(Y) \leq \omega$, then $X \times Y$ is acc.

Further, Bonanzinga [1] proved that

(iii) if X is an hacc T_3-space and Y is a sequential, compact T_2-space, then $X \times Y$ is hacc, and

(iv) if X is an ω-bounded, hacc T_3-space and Y is a compact T_2-space with $t(Y) \leq \omega$, then $X \times Y$ is hacc.

In Section 2, we show that Vaughan's theorems (i), (ii) and Bonanzinga's theorems (iii), (iv) are deduced from Theorem 1. Matveev [8] asked if there exists a Tychonoff space which has not the property (wa). In Section 3, we answer the question by proving the following theorem:

Theorem 2. *There exists a* 0*-dimensional, first countable, Tychonoff space without the property* (*wa*)*.*

Matveev [9] also asked if there exists a separable, countably compact, topological group which is not acc. Vaughan [11] asked the same question and showed that the answer is positive if there is a separable, sequentially compact T_2-group which is not compact. From this point of view, he also asked if there is a separable, sequentially compact T_2-group which is not compact. The final theorem below, which is a joint work with Ohta, answers the questions. Let $\mathfrak{s}$ denote the splitting number, i.e., $\mathfrak{s} = min\{\kappa : \text{the power } 2^\kappa \text{ is not sequentially compact }\}$ (cf. [2 Theorem 6.1]).

Theorem 3. (Ohta-Song). *There exists a separable, countably compact T_2-group which is not acc. If $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$, then there exists a separable, sequentially compact T_2-group which is not acc.*

It was shown in the proof [2 Theorem 5.4] that the assumption that $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$ is consistent with ZFC. Theorem 3 will be proved in Section 4.

Remark 1. Theorem 2 was proved independently by Just, Matveev and Szeptycki [5]. Matveev kindly informed Ohta that a similar theorem to Theorem 3 above was also proved independently by W. Pack in his Ph. D thesis at the University of Oxford (1997).

For a set A, $|A|$ denotes the cardinality of A. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. Other terms and symbols will be used as in [3].

§2. Theorem 1 and its corollaries

Throughtout this section, κ stands for an infinite cardinal. For a set A, let $[A]^{\leq\kappa} = \{B : B \subseteq A, |B| \leq \kappa\}$ and $[A]^{<\kappa} = \{B : B \subseteq A, |B| < \kappa\}$. For a subset A of a space X, we define the *κ-closure* of A in X by $\kappa\text{-cl}_X A = \cup\{\text{cl}_X B : B \in [A]^{\leq\kappa}\}$ and say that A is *κ-closed* in X if $A = \kappa\text{-cl}_X A$. By the definition, $\kappa\text{-cl}_X A$ is always κ-closed in X.

Lemma 4. *Let X be a space. Then, $t(X) \leq \kappa$ if and only if every κ-closed set in X is closed.*

Lemma 5. *Let X and Y be spaces such that $\pi_Y : X \times Y \to Y$ is closed map. Then, $\pi_Y(A)$ is κ-closed in Y for each κ-closed set A in $X \times Y$.*

Theorem 1 will be proved by using Lemmas 4 and 5. We now proceed to corollaries. The first one follows immediately from Theorem 1:

Corollary 6. *Let X be an initially κ-compact, acc (resp. hacc) T_3-space and Y a compact T_2-space with $t(Y) \leq \kappa$. Assume that $\pi_Y : X \times Y \to Y$ is a closed map. Then, $X \times Y$ is acc (resp. hacc).*

Since an acc space is countably compact (i.e., initially ω-compact), the following corollary is a special case of the preceding corollary.

Corollary 7. *Let X be an acc (resp. hacc) T_3-space and Y a compact T_2-space with $t(Y) \leq \omega$. Assume $\pi_Y : X \times Y \to Y$ is a closed map. Then, $X \times Y$ is acc (resp. hacc).*

It is known (cf, [3, Theorem 3.10.7]) that if X is countably compact and Y is sequential, then $\pi_Y : X \times Y \to Y$is closed. Hence, we have the following corollary, which is Vaughan's theorem (i) and Bonanzinga's theorem (iii) stated in the introduction:

Corollary 8. (Vaughan [12] and Bonanzinga [1]) *Let X be an acc (resp. hacc) T_3-space and Y a sequential, compact T_2-space, Then, $X \times Y$ is acc (resp. hacc).*

Recall that a space X is *κ-bounded* if $\mathrm{cl}_X A$ is compact for each $A \in [X]^{\leq\kappa}$. It is known (cf. [10]) that all κ-bounded spaces are initially κ-compact. Further, Kombarov [6] proved that if X is κ-bounded and $t(Y) \leq \kappa$, then $\pi_Y : X \times Y \to Y$ is closed. Hence, we have the following corollary, which generalizes Vaughan's theorem (ii) and Bonanzinga's theorem (iv) stated in the introduction.

Corollary 9. *Let X be a κ-bounded, acc (resp. hacc) T_3-space and Y a compact T_2-space with $t(Y) \leq \kappa$, then $X \times Y$ is acc (resp. hacc).*

§ 3. Proof of Theorem 2

In this section, we give a proof of Theorem 2. We omit a simple proof of the following lemma.

Lemma 10. *Let $\mathbb{R}$ be the space of real numbers with the usual topology and A a discrete subspace of $\mathbb{R}$. Then, $|A| \leq \omega$ and $\mathrm{cl}_{\mathbb{R}} A$ is nowhere dense in $\mathbb{R}$.*

Proof of Theorem 2. Let $A = \bigcup_{n \in N} A_n$, where $A_n = \mathbb{Q} \times \{1/n\}$ and let $\mathcal{A} = \{S : S$ is a discrete subspace of $A\}$. Then, we have:

Claim 1. $|\mathcal{A}| = \mathfrak{c}$.

Proof. Since $|A| \leq \omega$, $|\mathcal{A}| \leq \mathfrak{c}$. Let $S = \{\langle n, 1\rangle : n \in N\} \subseteq A$. Since every subset of S is discrete, $\{F : F \subseteq S\} \subseteq \mathcal{A}$. Hence, $|\mathcal{A}| \geq |\{F : F \subseteq S\}| = \mathfrak{c}$. □

Since $|\mathcal{A}| = \mathfrak{c}$, we can enumerate the family $\mathcal{A}$ as $\{S_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$ and each $n \in N$, put $S_{\alpha,n} = \{q \in \mathbb{Q} : \langle q, 1/n\rangle \in S_\alpha\}$.

Claim 2. *For each* $\alpha < \mathfrak{c}$, $|\mathbb{R} \setminus \bigcup_{n\in N} \mathrm{cl}_{\mathbb{R}} S_{\alpha,n}| = \mathfrak{c}$.

Proof. For each $\alpha < \mathfrak{c}$, let $X_\alpha = \mathbb{R} \setminus \bigcup_{n\in N} \mathrm{cl}_{\mathbb{R}} S_{\alpha,n}$. Since X_α is a G_δ-set in $\mathbb{R}$, X_α is a complete metric space. To show that X_α is dense in itself, suppose that X_α has an isolated point x. Then, there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \cap X_\alpha = \{x\}$. Let $I = (x, x+\varepsilon)$, Then, $I \subset \mathbb{R} \setminus X_\alpha \subset \bigcup_{n\in N} \mathrm{cl}_{\mathbb{R}} S_{\alpha,n}$. Moreover, since I is open in $\mathbb{R}$, $\mathrm{cl}_{\mathbb{R}} S_{\alpha,n} \cap I \subseteq \mathrm{cl}_{\mathbb{R}}(S_{\alpha,n} \cap I)$. Hence,

$$I = \Big(\bigcup_{n\in N} \mathrm{cl}_{\mathbb{R}} S_{\alpha,n}\Big) \cap I = \bigcup_{n\in N} (\mathrm{cl}_{\mathbb{R}} S_{\alpha,n} \cap I) \subseteq \bigcup_{n\in N} \mathrm{cl}_{\mathbb{R}}(S_{\alpha,n} \cap I). \tag{6}$$

By Lemma 10, each $\mathrm{cl}_{\mathbb{R}}(S_{\alpha,n} \cap I)$ is nowhere dense in $\mathbb{R}$. Thus, (6) contradicts the Baire Category Theorem. Hence, X_α is dense in itself. It is known ([3, 4.5.5]) that every dense in itself complete metric space includes a Cantor set. Hence, $|X_\alpha| = \mathfrak{c}$. $\square$

Claim 3. *There exists a sequence* $\{p_\alpha : \alpha < \mathfrak{c}\}$ *satisfying the following conditions*:

(1) *For each* $\alpha < \mathfrak{c}$, $p_\alpha \in \mathbb{P}$.
(2) *For any* $\alpha, \beta < \mathfrak{c}$, *if* $\alpha \neq \beta$, *then* $p_\alpha \neq p_\beta$.
(3) *For each* $\alpha < \mathfrak{c}$, $p_\alpha \notin \bigcup_{n\in N} \mathrm{cl}_{\mathbb{R}} S_{\alpha,n}$.

Proof. By transfinite induction, we define a sequence $\{p_\alpha : \alpha < \mathfrak{c}\}$ as follows: There is $p_0 \in \mathbb{P}$ such that $p_0 \notin \bigcup_{n\in N} \mathrm{cl} S_{0,n}$ by Claim 2. Let $0 < \alpha < \mathfrak{c}$ and assume that p_β has been defined for all $\beta < \alpha$. By Claim 2, $|\mathbb{R} \setminus \bigcup_{n\in N} \mathrm{cl}_{\mathbb{R}} S_{\alpha,n}| = \mathfrak{c}$. Hence, we can choose a point $p_\alpha \in (\mathbb{P} \setminus \bigcup_{n\in N} \mathrm{cl}_{\mathbb{R}} S_{\alpha,n}) \setminus \{p_\beta : \beta < \alpha\}$. Now, we have completed the induction. Then, the sequence $\{p_\alpha : \alpha < \mathfrak{c}\}$ satisfies the conditions (1) (2) and (3). $\square$

Claim 4. *For each* $\alpha < \mathfrak{c}$, *there exists a sequence* $\{\varepsilon_{\alpha,n} : n \in N\}$ *in* $\mathbb{Q}$ *satisfying the following conditions:*

(1) *For each* $n \in N$, $(p_\alpha - \varepsilon_{\alpha,n}, p_\alpha + \varepsilon_{\alpha,n}) \cap S_{\alpha,n} = \emptyset$.
(2) *For each* $n \in N$, $\varepsilon_{\alpha,n} \geq \varepsilon_{\alpha,n+1}$.
(3) $\lim_{n\to\infty} \varepsilon_{\alpha,n} = 0$.

Proof. Let $\alpha < \mathfrak{c}$. For $n = 1$, since $p_\alpha \notin \mathrm{cl}_{\mathbb{R}} S_{\alpha,1}$, there exists a rational $\varepsilon_{\alpha,1} > 0$ such that $(p_\alpha - \varepsilon_{\alpha,1}, p_\alpha + \varepsilon_{\alpha,1}) \cap S_{\alpha,1} = \emptyset$. Let $n > 1$ and assume that we have defined $\{\varepsilon_{\alpha,m} : m < n\}$ satisfying that $\varepsilon_{\alpha,1} > \varepsilon_{\alpha,2} > \cdots > \varepsilon_{\alpha,n-1}$. Since $p_\alpha \notin \mathrm{cl}_{\mathbb{R}} S_{\alpha,n}$, there exists a rational $\varepsilon'_{\alpha,n}$ such that $(p_\alpha - \varepsilon'_{\alpha,n}, p_\alpha + \varepsilon'_{\alpha,n}) \cap S_{\alpha,n} = \emptyset$. Put $\varepsilon_{\alpha,n} = n^{-1} \min\{\varepsilon_{\alpha,n-1}, \varepsilon'_{\alpha,n}\}$. Now, we have completed the induction. Then, the sequence $\{\varepsilon_{\alpha,n} : n \in N\}$ satisfies (1) (2) and (3). $\square$

Define $X = A \cup B$, where $B = \{\langle p_\alpha, 0\rangle : \alpha < \mathfrak{c}\}$, Topologize X as follows: A basic neighborhood of a point in A is a neighborhood induced from the usual topology on the plane. For each $\alpha < \mathfrak{c}$, a basic neighborhood base $\{U_n\langle p_\alpha, 0\rangle : n \in \omega\}$ of $\langle p_\alpha, 0\rangle \in B$ is defined by

$$U_n\langle p_\alpha, 0\rangle = \{\langle p_\alpha, 0\rangle\} \cup \Big(\bigcup_{i\geq n} \{((p_\alpha - \varepsilon_{\alpha,i}, p_\alpha + \varepsilon_{\alpha,i}) \cap \mathbb{Q}) \times \{1/i\}\}\Big).$$

for each $n \in N$. Then, X is a first countable T_2-space. For each $\alpha < \mathfrak{c}$ and each $n \in N$. $U_n\langle p_\alpha, 0\rangle$ is open and closed in X, because $p_\alpha \pm \varepsilon_{\alpha,i} \notin \mathbb{Q}$ for each $i \in \omega$. It follows that X is 0-dimensional, and hence, a Tychonoff space.

Claim 5. *The space X has not the property* (wa).

Proof. Let $\mathcal{U} = \{A\} \cup \{U_1\langle p_\alpha, 0\rangle : \alpha < \mathfrak{c}\}$. Then, $\mathcal{U}$ is an open cover of X and A is a dense subspace of X. For each discrete subset F of A, there exists $\alpha < \mathfrak{c}$ such that $F = S_\alpha$. Since $U_1\langle p_\alpha, 0\rangle \cap S_\alpha = \emptyset$, $\langle p_\alpha, 0\rangle \notin \mathrm{St}(F, \mathcal{U})$. This shows that X does not have the property (wa). □

§4. Proof of Theorem 3

We omit the proofs of the following lemmas and only show how Theorem 3 can be deduced from the lemmas.

Lemma 11. *Let X be a space and Y a space having at least one pair of disjoint nonempty closed subsets. Assume that $X \times Y^\kappa$ is acc for an infinite cardinal κ. Then, X is initially κ-compact.*

We consider $2 = \{0,1\}$ the discrete group of integers modulo 2. Then, 2^κ is a topological group under pairwise addition. The following lemma seems to be well known (see [10, 3.5] for the first statement), but we include it here for the sake of completeness.

Lemma 12. *There exists a separable, countably compact, non-compact subgroup G_1 of $2^{\mathfrak{c}}$. If $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$, then there exists a separable, sequentially compact, non-compact subgroup G_2 of 2^{ω_1}.*

Proof of Theorem 3. Let G_1 be the group in Lemma 12. Then, $G_1 \times 2^{\mathfrak{c}}$ is a separable, countably compact T_2-group. Since G_1 is not compact and $w(G_1) \leq \mathfrak{c}$, G_1 is not initially $\mathfrak{c}$-compact. Hence, it follows from Lemma 11 that $G_1 \times 2^{\mathfrak{c}}$ is not acc. Next, assume that $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$, and let G_2 be the group in Lemma 12. Since $\omega_1 < \mathfrak{s}$, 2^{ω_1} is sequentially compact. Hence, $G_2 \times 2^{\omega_1}$ is a separable, sequentially compact T_2-group which is not compact. Since $w(G_2) = \omega_1$, $G_2 \times 2^\omega$ is not acc by Lemma 11. □

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